

DEFORMATIONS OF PRODUCT-QUOTIENT SURFACES AND RECONSTRUCTION OF TODOROV SURFACES VIA \mathbb{Q} -GORENSTEIN SMOOTHING

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ABSTRACT. We consider the deformation spaces of some singular product-quotient surfaces $X = (C_1 \times C_2)/G$, where the curves C_i have genus 3 and the group G is isomorphic to \mathbb{Z}_4 . As a by-product, we give a new construction of Todorov surfaces with $p_g = 1$, $q = 0$ and $2 \leq K^2 \leq 8$ by using \mathbb{Q} -Gorenstein smoothings.

0. INTRODUCTION

In [To81], Todorov constructed some surfaces of general type with $p_g = 1$, $q = 0$ and $2 \leq K^2 \leq 8$ in order to give counterexamples of the global Torelli theorem. Todorov surfaces with $K^2 = 8 - k$ are double covers of a Kummer surface in \mathbb{P}^3 branched over a curve D , which is a complete intersection of the Kummer surface with a smooth quadric surface containing k of its nodes, and over the remaining $16 - k$ nodes. Surfaces with $K^2 = 2$, and $p_g = 1$ have been completely classified by Catanese and Debarre [CD89], while some examples were constructed by Todorov. C. Rito [Rito09] gave a detailed study of Todorov surfaces with an involution.

Recently, H. Park, J. Park and D. Shin constructed simply connected surfaces of general type with $p_g = 1$, $q = 0$ and $2 \leq K^2 \leq 8$ by considering \mathbb{Q} -Gorenstein smoothings of singular K3 surfaces with special configurations of cyclic quotient singularities, see [PPS1], [PPS2]. Their construction follows the method used by Lee and Park in the paper [LP07], where a simply connected surface of general type with $p_g = q = 0$ and $K^2 = 2$ is constructed via the \mathbb{Q} -Gorenstein smoothing of a singular rational surface. For more details about these kind of techniques, over a field of any characteristic, we refer the reader to the work of Lee and Nakayama [LN11].

Moreover, Bauer, Catanese, Grunewald and Pignatelli constructed many interesting examples of surfaces of general type with $p_g = 0$ by considering the minimal desingularization of singular product-quotient surfaces, see [BC04], [BCG08], [BCGP], [BP]. Similar methods are applied to surfaces of general type with $p_g = q = 1$ by Polizzi and others, see [Pol08], [Pol09], [CP09], [MP10]. These results motivated us to start the investigation of \mathbb{Q} -Gorenstein smoothings of singular product-quotient surfaces.

Let us recall that a projective surface S is called a *product-quotient surface* if there exists a finite group G , acting faithfully on two smooth curves C_1 and C_2 and diagonally on their product, so that S is isomorphic to the minimal

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desingularization of $X = (C_1 \times C_2)/G$. The surface X is called a *singular model of a product-quotient surface*, or simply a *singular product-quotient surface*.

This paper focuses on the case $g(C_1) = g(C_2) = 3$ and $G = \mathbb{Z}_4$. More precisely, we assume that there exist two simple \mathbb{Z}_4 -covers $g_i: C_i \rightarrow \mathbb{P}^1$, both branched in four points. Then the singular product-quotient surface

$$X := (C_1 \times C_2)/\mathbb{Z}_4$$

contains precisely 16 cyclic quotient singularities; any of them is either of type $\frac{1}{4}(1, 1)$ or of type $\frac{1}{4}(1, 3)$. Note that $\frac{1}{4}(1, 3)$ is a rational double point, whereas $\frac{1}{4}(1, 1)$ is a singularity of class T , so both admit a *local* \mathbb{Q} -Gorenstein smoothing, see [KSB88] or [Man08, Sections 2-4]. The problem is to understand whether these local smoothings can be glued together in order to have a *global* \mathbb{Q} -Gorenstein smoothing of X . We will show that in some cases this is actually possible.

This paper is organized as follows.

In Section 1 we present some preliminaries and we set up notation and terminology. In particular, we recall the definitions of simple cyclic cover of a curve and of singular product-quotient surface and we explain how to compute their basic invariants.

In Section 2 we introduce the main objects that we want to study, namely the singular product quotient surfaces of the form $X = (C_1 \times C_2)/G$, where $g(C_1) = g(C_2) = 3$, $G = \mathbb{Z}_4$ and $C_i \rightarrow C_i/G$ is a simple cyclic cover for $i = 1, 2$.

Section 3 deals with the study of the singular product-quotient surface $Y = (C_1 \times C_2)/H$, where H is the unique subgroup of G isomorphic to \mathbb{Z}_2 . By construction, Y contains exactly 16 ordinary double points as singularities. By using the infinitesimal techniques introduced in [Pin81] and [Cat89], we prove that $\text{Def}(Y)$ is smooth at Y , of dimension 18 and $\text{ESDef}(Y)$ is smooth at $[Y]$, of dimension 8 (Proposition 3.6). Moreover, if $\mu: V \rightarrow Y$ is the minimal desingularization of Y , we have

$$\dim_{[V]} \text{Def}(V) = 18, \quad h^1(\Theta_V) = 24,$$

hence $\text{Def}(V)$ is singular at $[V]$; by [BW74] this implies that the sixteen (-2) curves of V do not have independent behavior in deformations.

In Section 4 we discuss three examples of singular product-quotient surface $X = (C_1 \times C_2)/G$ with different G -action.

- In the first example we have $\text{Sing}(X) = 16 \times \frac{1}{4}(1, 3)$, so X contains only rational double points as singularities. We prove that $\text{Def}(X)$ and $\text{ESDef}(X)$ are both smooth at $[X]$, of dimension 44 and 2, respectively (Propositions 4.4 and 4.2).

The surface X satisfies $h^0(\omega_X) = 5$ and $K_X^2 = 8$; moreover it is not difficult to see that the canonical map $\phi_K: X \rightarrow \mathbb{P}^4$ is a birational morphism onto its image; by [Cat97, Proposition 6.2] it follows that the general deformation of X is isomorphic to a smooth complete intersection of bidegree $(2, 4)$ in \mathbb{P}^4 .

Moreover we have

$$\dim_{[S]} \text{Def}(S) = 44, \quad h^1(\Theta_S) = 50,$$

hence $\text{Def}(S)$ is singular at S . This means that the sixteen A_3 -cycles of S do not have independent behavior in deformations.

- In the second example we have $\text{Sing}(X) = 16 \times \frac{1}{4}(1, 1)$. We show that there exist a \mathbb{Q} -Gorenstein smoothing $\pi: \mathcal{X} \rightarrow T$ of X , whose base T has dimension 12, such that the general fibre X_t of π is a minimal surface of general type whose invariants are

$$p_g(X_t) = 1, \quad q(X_t) = 0, \quad K_{X_t}^2 = 8.$$

Moreover X_t is isomorphic to a Todorov surface with $K^2 = 8$ (Theorem 4.6). By a slight modification of the construction, it is possible to obtain all Todorov surfaces with $2 \leq K^2 \leq 8$.

This is related to the existence of complex structures on rational blow-downs of algebraic surfaces. More precisely, one can consider the rational blow-down $S(t)$ of t of the (-4) -curves in S , where $1 \leq t \leq 16$. This means that one considers the normal connected sum of S with t copies of \mathbb{P}^2 , identifying a conic in each \mathbb{P}^2 with a (-4) -curve in S ; then $S(t)$ is a symplectic 4-manifold. One can therefore raise the following:

Question. Is it possible to give a complex structure on $S(t)$ for $1 \leq t \leq 16$, and to describe $S(t)$ when such a complex structure exists?

Our results answer affirmatively this question when $10 \leq t \leq 16$; in these cases, indeed, one can give a complex structure to the rational blow-down $S(t)$, which make it isomorphic to a Todorov surface with $K^2 = t - 8$.

- In the third example, we have $\text{Sing}(X) = 8 \times \frac{1}{4}(1, 1) + 8 \times \frac{1}{4}(1, 3)$. Rasdeaconu and Suvaina give an explicit construction of the minimal desingularization S of X , see [RS06, Section 3]; in fact, they prove that S is a simply connected, minimal elliptic surface with no multiple fibres.

We show that there exists a \mathbb{Q} -Gorenstein smoothing of X , although $H^2(\Theta_X) \neq 0$ and all the natural deformations of the G -cover $u: X \rightarrow Q$ preserve the 8 singularities of type $\frac{1}{4}(1, 1)$, see Proposition 4.8. Indeed we prove that a general surface \bar{X} in the subfamily of natural deformations of the G -cover of X can be deformed to a bidouble cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched over three smooth divisors of bidegree $(2, 2)$. By taking a general deformation of these three divisors we obtain a \mathbb{Q} -Gorenstein smoothing of X which smoothes all the singularities. More generally, by using the same method one can construct surfaces of general type with $p_g = 3$, $q = 0$ and $K^2 = k$ ($2 \leq k \leq 8$) by first taking a \mathbb{Q} -Gorenstein smoothing of k singular points of type $\frac{1}{4}(1, 1)$ of \bar{X} and then the minimal resolution of the remaining $8 - k$ singular points of the same type.

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Notation and conventions.

We work over the field \mathbb{C} of complex numbers.

By “surface” we mean a projective, non-singular surface S , and for such a surface $\omega_S = \mathcal{O}_S(K_S)$ denotes the canonical class, $p_g(S) = h^0(S, \omega_S)$ is the *geometric genus*, $q(S) = h^1(S, \omega_S)$ is the *irregularity* and $\chi(\mathcal{O}_S) = 1 - q(S) + p_g(S)$ is the *Euler-Poincaré characteristic*.

If X is any (possibly singular) projective scheme, we denote by $\text{Def}(X)$ the base of the Kuranishi family of deformations of X and by $\text{ESDef}(X)$ the base of the equisingular deformations of X . The tangent spaces to $\text{Def}(X)$ and $\text{ESDef}(X)$ at the point $[X]$ corresponding to X are given by $\text{Ext}^1(\Omega_Y^1, \mathcal{O}_Y)$ and $H^1(\Theta_Y)$, respectively.

If L is a line bundle L on X , we use the notation L^n instead of $L^{\otimes n}$ if no confusion can arise.

If G is any finite abelian group, we denote by \widehat{G} its dual group, namely the group of irreducible characters of G .

1. PRELIMINARIES

1.1. Simple cyclic covers of curves. Let Γ be a smooth, projective curve and $B \subset \Gamma$ an effective divisor such that $\mathcal{O}_\Gamma(B) = \mathcal{L}^n$ for some $\mathcal{L} \in \text{Pic}(\Gamma)$. Therefore there exists a \mathbb{Z}_n -cover $g: C \rightarrow \Gamma$, totally branched over B , which is called a *simple cyclic cover*. We identify \mathbb{Z}_n with the group of n -th roots of unity, namely $\mathbb{Z}_n = \langle \zeta \rangle$, where ζ is a primitive n -th root. The dual group $\widehat{\mathbb{Z}_n}$ is isomorphic to \mathbb{Z}_n , and it is generated by the character $\chi_1: \mathbb{Z}_n \rightarrow \mathbb{C}$ such that $\chi_1(\zeta) = \zeta^{-1}$. We will write χ_j instead of χ_1^j ; then $\chi_j(\zeta) = \zeta^{-j}$. The group \mathbb{Z}_n acts naturally on $g_*\mathcal{O}_C$, so there is a canonical splitting

$$(1) \quad g_*\mathcal{O}_C = \mathcal{O}_\Gamma \oplus \mathcal{L}^{-1} \oplus \dots \oplus \mathcal{L}^{-(n-1)},$$

where the summand \mathcal{L}^{-j} is the eigensheaf $(g_*\mathcal{O}_C)^{\chi_j}$ corresponding to the character χ_j .

Similarly, \mathbb{Z}_n acts naturally on $g_*\omega_C$ and $g_*\omega_C^2$, giving the following decompositions (see [Pa91] and [Cat89, Section 2]):

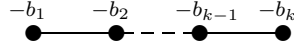
$$(2) \quad \begin{aligned} g_*\omega_C &= \omega_\Gamma \oplus (\omega_\Gamma \otimes \mathcal{L}) \oplus \dots \oplus (\omega_\Gamma \otimes \mathcal{L}^{n-1}), \\ g_*\omega_C^2 &= (\omega_\Gamma^2(B) \otimes \mathcal{L}^{-1}) \oplus \omega_\Gamma^2(B) \oplus \dots \oplus (\omega_\Gamma^2(B) \otimes \mathcal{L}^{n-2}). \end{aligned}$$

In the equations (2), the eigensheaves corresponding to χ_j are $\omega_\Gamma \otimes \mathcal{L}^j$ and $\omega_\Gamma^2(B) \otimes \mathcal{L}^j$, respectively.

1.2. Cyclic quotient singularities, Hirzebruch Jung resolutions and singular product-quotient surfaces. Let n and q be natural numbers with $0 < q < n$, $(n, q) = 1$ and let ζ be a primitive n -th root of unity. Let us consider the action of the cyclic group $\mathbb{Z}_n = \langle \zeta \rangle$ on \mathbb{C}^2 defined by $\zeta \cdot (x, y) = (\zeta x, \zeta^q y)$. Then the analytic space $X_{n,q} = \mathbb{C}^2 / \mathbb{Z}_n$ has a cyclic quotient singularity of type $\frac{1}{n}(1, q)$, and $X_{n,q} \cong X_{n',q'}$ if and only if $n = n'$ and either $q = q'$ or $qq' \equiv 1 \pmod{n}$. The exceptional divisor on the minimal resolution $\tilde{X}_{n,q}$ of $X_{n,q}$ is a Hirzebruch-Jung string, that is to say, a connected union $E = \bigcup_{i=1}^k Z_i$ of smooth rational curves Z_1, \dots, Z_k with self-intersection ≤ -2 , and ordered linearly so that $Z_i Z_{i+1} = 1$ for all i , and $Z_i Z_j = 0$ if $|i - j| \geq 2$. More precisely, given the continued fraction

$$\frac{n}{q} = [b_1, \dots, b_k] = b_1 - \frac{1}{b_2 - \frac{1}{\dots - \frac{1}{b_k}}}, \quad b_i \geq 2,$$

the dual graph of E is



(cf. [Lau71, Chapter II]). Notice that a rational double point of type A_n corresponds to the cyclic quotient singularity $\frac{1}{n+1}(1, n)$.

Definition 1.1. Let x be a cyclic quotient singularity of type $\frac{1}{n}(1, q)$. Then we set

$$\begin{aligned} \mathfrak{h}_x &= 2 - \frac{2 + q + q'}{n} - \sum_{i=1}^k (b_i - 2), \\ \mathfrak{e}_x &= k + 1 - \frac{1}{n}, \\ B_x &= 2\mathfrak{e}_x - \mathfrak{h}_x = \frac{1}{n}(q + q') + \sum_{i=1}^k b_i, \end{aligned}$$

where $1 \leq q' \leq n - 1$ is such that $qq' \equiv 1 \pmod{n}$.

Definition 1.2. [BP] We say that a projective surface S is a product-quotient surface if there exists a finite group G acting faithfully on two smooth projective curves C_1 and C_2 and diagonally on their product, so that S is isomorphic to the minimal desingularization of $X := (C_1 \times C_2)/G$. The surface X is called a singular model of a product-quotient surface, or simply a singular product-quotient surface.

From this definition it follows that a singular product quotient surface contains a finite number of cyclic quotient singularities.

Proposition 1.3 (cf. [MP10], Section 3). Let S be a product quotient surface, minimal desingularization of $X = (C_1 \times C_2)/G$. Then the invariants of S are

$$(i) \quad K_S^2 = \frac{8(g(C_1)-1)(g(C_2)-1)}{|G|} + \sum_{x \in \text{Sing } X} \mathfrak{h}_x.$$

$$(ii) \quad e(S) = \frac{4(g(C_1)-1)(g(C_2)-1)}{|G|} + \sum_{x \in \text{Sing } X} \epsilon_x.$$

$$(iii) \quad q(S) = g(C_1/G) + g(C_2/G).$$

Set $\Gamma_i := C_i/G$ and let $g_i: C_i \rightarrow \Gamma_i$. The group G acts naturally on the sheaves $g_{i*}\mathcal{O}_{C_i}$, $g_{i*}\omega_{C_i}$, $g_{i*}\omega_{C_i}^2$. Assuming that G is *abelian*, we can write the following generalizations of (1) and (2):

$$\begin{aligned} g_{i*}\mathcal{O}_{C_i} &= \bigoplus_{\chi \in \widehat{G}} (g_{i*}\mathcal{O}_{C_i})^\chi, \\ g_{i*}\omega_{C_i} &= \bigoplus_{\chi \in \widehat{G}} (g_{i*}\omega_{C_i})^\chi, \\ g_{i*}\omega_{C_i}^2 &= \bigoplus_{\chi \in \widehat{G}} (g_{i*}\omega_{C_i}^2)^\chi, \end{aligned}$$

where $(*)^\chi$ is the eigensheaf corresponding to the character $\chi \in \widehat{G}$.

2. THE MAIN CONSTRUCTION

Let us consider two smooth curves C_1, C_2 of genus 3, such that there are two *simple* \mathbb{Z}_4 -covers $g_i: C_i \rightarrow \mathbb{P}^1$, both branched in 4 points. In the rest of the paper we write $G := \mathbb{Z}_4 = \langle \zeta \mid \zeta^4 = 1 \rangle$, where ζ is a primitive fourth root of unity; we also denote by H the subgroup of G defined by $H := \langle \zeta^2 \rangle \cong \mathbb{Z}_2$.

Now set $Z := C_1 \times C_2$ and consider the singular product-quotient surface

$$(3) \quad X := Z/G,$$

which has exactly 16 isolated singular points, corresponding to the fixed points of the G -action on Z . Let $\lambda: S \rightarrow X$ be the minimal resolution of singularities of X .

The G -cover g_i factors through the double cover $h_i: C_i \rightarrow E_i$, where $E_i := C_i/H$. Note that E_i is an elliptic curve and that the singular product-quotient surface

$$(4) \quad Y := Z/H$$

contains sixteen cyclic quotient singularities of type $\frac{1}{2}(1, 1)$, i.e. ordinary double points, as only singularities. Let us denote by $\mu: V \rightarrow Y$ the minimal desingularization of Y . We have a commutative diagram

$$(5) \quad \begin{array}{ccccc} V & \xrightarrow{\mu} & Y & \xrightarrow{v} & E_1 \times E_2, \\ & & \swarrow r & & \uparrow h \\ & & Z & & \\ & & \swarrow p & & \downarrow t \\ S & \xrightarrow{\lambda} & X & \xrightarrow{u} & \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

where:

- $p: Z \rightarrow X$ and $r: Z \rightarrow Y$ are the natural projections, so $s: Y \rightarrow X$ is a double cover (more precisely, a G/H -cover) branched over the singular points of X ;

- $g := g_1 \times g_2: Z \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is a $G \times G$ -cover branched on a divisor $B \subset \mathbb{P}^1 \times \mathbb{P}^1$ of product type and of bidegree $(4, 4)$;
- $h := h_1 \times h_2: Z \rightarrow E_1 \times E_2$ is a $H \times H$ -cover branched on a divisor $\Delta \subset E_1 \times E_2$ of product type and of bidegree $(4, 4)$;
- $u: X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is a G -cover, whose branch locus coincides with B ;
- $v: Y \rightarrow E_1 \times E_2$ is a H -cover, whose branch locus coincides with Δ ;
- $t: E_1 \times E_2 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is a $G/H \times G/H$ -cover whose branch locus is B and whose ramification locus is Δ .

Let us denote by B_i the branch locus of $g_i: C_i \rightarrow \mathbb{P}^1$ and by Δ_i the branch locus of $h_i: C_i \rightarrow E_i$. Both B_i and Δ_i consist of four points; clearly $B = B_1 \times B_2$ and $\Delta = \Delta_1 \times \Delta_2$. From the results of Section 1 we infer that

- there is a natural action of G on the sheaves $g_{i*}\mathcal{O}_{C_i}$, $g_{i*}\omega_{C_i}$, $g_{i*}\omega_{C_i}^2$, which gives decompositions:

$$\begin{aligned}
 (6) \quad & g_{i*}\mathcal{O}_{C_i} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{M}_i^{-1} \oplus \mathcal{M}_i^{-2} \oplus \mathcal{M}_i^{-3}; \\
 & g_{i*}\omega_{C_i} = \omega_{\mathbb{P}^1} \oplus (\omega_{\mathbb{P}^1} \otimes \mathcal{M}_i) \oplus (\omega_{\mathbb{P}^1} \otimes \mathcal{M}_i^2) \oplus (\omega_{\mathbb{P}^1} \otimes \mathcal{M}_i^3); \\
 & g_{i*}\omega_{C_i}^2 = \omega_{\mathbb{P}^1}^2(B_i) \oplus (\omega_{\mathbb{P}^1}^2(B_i) \otimes \mathcal{M}_i) \oplus (\omega_{\mathbb{P}^1}^2(B_i) \otimes \mathcal{M}_i^2) \\
 & \quad \oplus (\omega_{\mathbb{P}^1}^2(B_i) \otimes \mathcal{M}_i^{-1}),
 \end{aligned}$$

where $\mathcal{M}_i = \mathcal{O}_{\mathbb{P}^1}(1)$. Left to right, the direct summands are the four eigensheaves corresponding to the four characters $\chi_0, \chi_1, \chi_2, \chi_3$ of G ;

- there is a natural action of H on the sheaves $h_{i*}\mathcal{O}_{C_i}$, $h_{i*}\omega_{C_i}$, $h_{i*}\omega_{C_i}^2$, which gives decompositions:

$$\begin{aligned}
 (7) \quad & h_{i*}\mathcal{O}_{C_i} = \mathcal{O}_{E_i} \oplus \mathcal{L}_i^{-1}, \\
 & h_{i*}\omega_{C_i} = \omega_{E_i} \oplus (\omega_{E_i} \otimes \mathcal{L}_i), \\
 & h_{i*}\omega_{C_i}^2 = \omega_{E_i}^2(\Delta_i) \oplus (\omega_{E_i}^2(\Delta_i) \otimes \mathcal{L}_i^{-1}),
 \end{aligned}$$

where \mathcal{L}_i is a line bundle of degree 2 on C_i such that $\mathcal{L}_i^2 = \mathcal{O}_{E_i}(\Delta_i)$. Left to right, the direct summands correspond to the invariant and anti-invariant eigensheaves for the H -action, respectively.

3. DEFORMATIONS OF THE SINGULAR PRODUCT-QUOTIENT SURFACE

$$Y = Z/H$$

Let us consider again the surface $Y = Z/H$ defined in Section 2, together with its minimal desingularization $\mu: V \rightarrow Y$. As we remarked in the previous section, we have

$$\text{Sing}(Y) = 16 \times \frac{1}{2}(1, 1).$$

Proposition 3.1. *V is a minimal surface of general type whose invariants are*

$$\begin{aligned}
 p_g(V) &= 5, & q(V) &= 2, & K_V^2 &= 16, \\
 h^1(\Theta_V) &= 24, & h^2(\Theta_V) &= 16.
 \end{aligned}$$

Proof. The invariants $p_g(V)$, $q(V)$, K_V^2 can be computed by using Proposition 1.3. Since $p_g(V) > 0$ and $K_V^2 > 0$, it follows that V is a surface of general type. Let us denote by $H^0(*)^+$ and $H^0(*)^-$ the spaces of invariant

and anti-invariant sections for the H -action and by $h^0(*)^+$ and $h^0(*)^-$ their dimensions. Since Y has only rational double points, Künneth formula and the third equality in (7) give

$$\begin{aligned} H^0(\omega_V^2) &= H^0(\omega_Y^2) = H^0(\omega_Z^2)^+ = H^0(\omega_{C_1}^2 \boxtimes \omega_{C_2}^2)^+ \\ &= (H^0(h_{1*}\omega_{C_1}^2)^+ \otimes H^0(h_{2*}\omega_{C_2}^2)^+) \oplus (H^0(h_{1*}\omega_{C_1}^2)^- \otimes H^0(h_{2*}\omega_{C_2}^2)^-) \\ &\cong \mathbb{C}^{20}. \end{aligned}$$

This shows that $h^0(\omega_V^2) = K_V^2 + \chi(\mathcal{O}_V)$, hence V is a minimal model.

Since Y is a normal surface, [BW74, Proposition 1.2] gives $\mu_*\Theta_V = \Theta_Y$. Therefore the argument in [BW74, Section 1] or [Cat89, p. 299] shows that there are two isomorphisms

$$(8) \quad H^1(\Theta_V) \cong H^1(\Theta_Y) \oplus H_E^1(\Theta_V), \quad H^2(\Theta_V) \cong H^2(\Theta_Y),$$

where $H_E^1(\Theta_V)$ denotes the local cohomology with support on the exceptional divisor $E \subset V$.

By the second isomorphism in (8), we have

$$(9) \quad H^2(\Theta_V)^* \cong H^2(\Theta_Y)^* = H^0(\Omega_Z^1 \otimes \Omega_Z^2)^+ = T_1 \oplus T_2 \oplus T_3 \oplus T_4,$$

where

$$\begin{aligned} T_1 &= H^0(h_{1*}\omega_{C_1}^2)^+ \otimes H^0(h_{2*}\omega_{C_2}^2)^+ = H^0(\omega_{E_1}^2(\Delta_1)) \otimes H^0(\omega_{E_2}), \\ T_2 &= H^0(h_{1*}\omega_{C_1}^2)^+ \otimes H^0(h_{2*}\omega_{C_2}^2)^+ = H^0(\omega_{E_1}) \otimes H^0(\omega_{E_2}^2(\Delta_2)), \\ T_3 &= H^0(h_{1*}\omega_{C_1}^2)^- \otimes H^0(h_{2*}\omega_{C_2}^2)^- \\ (10) \quad &= H^0(\omega_{E_1}^2(\Delta_1) \otimes \mathcal{L}_1^{-1}) \otimes H^0(\omega_{E_2} \otimes \mathcal{L}_2), \\ T_4 &= H^0(h_{1*}\omega_{C_1}^2)^- \otimes H^0(h_{2*}\omega_{C_2}^2)^- \\ &= H^0(\omega_{E_1} \otimes \mathcal{L}_1) \otimes H^0(\omega_{E_2}^2(\Delta_2) \otimes \mathcal{L}_2^{-1}). \end{aligned}$$

Since $\dim T_i = 4$ for all $i \in \{1, 2, 3, 4\}$, we infer $h^2(\Theta_V) = h^2(\Theta_Y) = 16$. By Riemann-Roch we have $h^1(\Theta_V) - h^2(\Theta_V) = 10\chi(\mathcal{O}_V) - 2K_V^2 = 8$, so it follows $h^1(\Theta_V) = 24$. \square

Corollary 3.2. *We have*

$$h^1(\Theta_Y) = 8, \quad h^2(\Theta_Y) = 16.$$

Proof. Since $h^2(\Theta_Y) = h^2(\Theta_V)$, the first equality follows from Proposition 3.1. Furthermore, E is the disjoint union of sixteen (-2) -curves, hence [BW74, Section 1] implies $H_E^1(\Theta_V) \cong \mathbb{C}^{16}$. Using $h^1(\Theta_V) = 24$ and the first isomorphism in (8) we obtain $h^1(\Theta_Y) = 8$, which completes the proof. \square

By using the local-to-global spectral sequence of $\mathcal{E}xt$ -sheaves we obtain an exact sequence

$$(11) \quad 0 \rightarrow H^1(\Theta_Y) \rightarrow \text{Ext}^1(\Omega_Y^1, \mathcal{O}_Y) \rightarrow \mathcal{T}_Y^1 \xrightarrow{\text{oby}} H^2(\Theta_Y),$$

where $\mathcal{T}_Y^1 := H^0(\mathcal{E}xt^1(\Omega_Y^1, \mathcal{O}_Y))$. Notice that \mathcal{T}_Y^1 is a skyscraper sheaf supported on the sixteen nodes of Y , hence oby is a linear map

$$\text{oby} : \mathbb{C}^{16} \rightarrow \mathbb{C}^{16}.$$

Thus its kernel and its cokernel have the same dimension.

Remark 3.3. *The branch locus Δ of $v: Y \rightarrow E_1 \times E_2$ is a polarization of type $(4, 4)$ on the abelian surface $E_1 \times E_2$, in particular $h^0(\Delta) = 16$. Since polarized abelian surfaces form a 3-dimensional family, it follows that the deformation space $\text{Def}(Y)$ has dimension at least 18. Therefore we have*

$$\dim \text{Ext}^1(\Omega_Y^1, \mathcal{O}_Y) = \dim T_{[Y]} \text{Def}(Y) \geq \dim_{[Y]} \text{Def}(Y) \geq 18.$$

Proposition 3.4. *We have*

$$\dim \ker \text{ob}_Y = \dim \text{coker } \text{ob}_Y = 10.$$

Proof. Notice that Remark 3.3 only gives $\dim(\ker \text{ob}_Y) \geq 10$. In order to prove equality, we apply an argument used in [Cat89, Section 2].

Let us consider the dual map $\text{ob}_Y^*: H^2(\Theta_Y)^* \rightarrow (\mathcal{T}_Y^1)^*$. We set

$$\begin{aligned} \Delta_1 &= d'_1 + d'_2 + d'_3 + d'_4 \\ \Delta_2 &= d''_1 + d''_2 + d''_3 + d''_4 \end{aligned}$$

and we choose local coordinates (x, y) in Z vanishing at (d'_i, d''_j) . Then the action of H with respect to these coordinates is given by $(x, y) \rightarrow (-x, -y)$.

By [Cat89] we have an isomorphism $(\mathcal{T}_Y^1)^* = (r_*\Omega_Z^1)^+/\Omega_Y^1$, therefore ob_Y^* can be seen as a map

$$\text{ob}_Y^*: H^0(\Omega_Z^1 \otimes \Omega_Z^2)^+ \rightarrow (r_*\Omega_Z^1)^+/\Omega_Y^1.$$

Near any of the ordinary double points of Y , the sheaf $(r_*\Omega_Z^1)^+$ is locally generated by $x dx, x dy, y dx, y dy$, whereas Ω_Y^1 is locally generated by $d(x^2), d(xy), d(y^2)$; then $(r_*\Omega_Z^1)^+/\Omega_Y^1$ is locally generated by $x dy - y dx$, cf. [Cat89, Lemma 2.11].

Looking at (10) and making straightforward computations, one checks that

- the summand T_1 contributes expressions of type $\alpha_1 \beta_1 y dx \otimes (dx \wedge dy)$;
- the summand T_2 contributes expressions of type $\alpha_2 \beta_2 x dy \otimes (dx \wedge dy)$;
- the summand T_3 contributes expressions of type $\alpha_3 \beta_3 x dx \otimes (dx \wedge dy)$;
- the summand T_4 contributes expressions of type $\alpha_4 \beta_4 y dy \otimes (dx \wedge dy)$,

where $\alpha_i = \alpha_i(x^2)$ and $\beta_i = \beta_i(y^2)$ are pullbacks of local functions on E_i .

Since in the \mathcal{O}_Y -module $(r_*\Omega_Z^1)^+/\Omega_Y^1$ we have the relations

$$1/2(x dy - y dx) = x dy = -y dx \text{ and } x dx = y dy = 0,$$

it follows that the restriction of ob_Y^* to the subspace $T_3 \oplus T_4$ is zero, whereas the restriction of ob_Y^* to the subspace $T_1 \oplus T_2$ can be identified, up to a multiplicative constant, with the map

$$\begin{aligned} \phi: H^0(\omega_{E_1}^2(\Delta_1)) \oplus H^0(\omega_{E_2}^2(\Delta_2)) &\rightarrow \bigoplus_{i,j=1}^4 \mathbb{C}_{ij}, \\ \phi(\sigma \oplus \tau) &= \bigoplus_{i,j=1}^4 (\text{val}_{d'_i}(\sigma) - \text{val}_{d''_j}(\tau)). \end{aligned}$$

Here the valuation maps $\text{val}_{d'_i}$ and $\text{val}_{d''_j}$ are defined, as usual, by the short exact sequences

$$(12) \quad \begin{aligned} 0 \rightarrow H^0(\omega_{E_1}^2) \rightarrow H^0(\omega_{E_1}^2(\Delta_1)) &\xrightarrow{\oplus \text{val}_{d'_i}} H^0(N_{\Delta_1}) \cong \oplus_{i=1}^4 \mathbb{C}_i, \\ 0 \rightarrow H^0(\omega_{E_2}^2) \rightarrow H^0(\omega_{E_2}^2(\Delta_2)) &\xrightarrow{\oplus \text{val}_{d''_j}} H^0(N_{\Delta_2}) \cong \oplus_{j=1}^4 \mathbb{C}_j. \end{aligned}$$

Therefore we obtain

$$(13) \quad \begin{aligned} \ker \phi = \{ \sigma \oplus \tau \mid \text{val}_{d'_1}(\sigma) = \text{val}_{d'_2}(\sigma) = \text{val}_{d'_3}(\sigma) = \text{val}_{d'_4}(\sigma) \\ = \text{val}_{d''_1}(\tau) = \text{val}_{d''_2}(\tau) = \text{val}_{d''_3}(\tau) = \text{val}_{d''_4}(\tau) \}. \end{aligned}$$

As E_i is an elliptic curve, we have $\omega_{E_i}^2 = \omega_{E_i}$ and so (12) are the standard residue sequences for meromorphic 1-forms. By the Residue Theorem we get

$$\sum_{i=1}^4 \text{val}_{d'_i}(\sigma) = \sum_{j=1}^4 \text{val}_{d''_j}(\tau) = 0,$$

hence (13) implies that $\sigma \oplus \tau \in \ker \phi$ if and only if $\text{val}_{d'_i}(\sigma) = \text{val}_{d''_j}(\tau) = 0$ for all pairs (i, j) . This yields $\ker \phi = H^0(\omega_{E_1}^2) \oplus H^0(\omega_{E_2}^2) \cong \mathbb{C} \oplus \mathbb{C}$.

Then $\ker \text{ob}^* = \ker \phi \oplus T_3 \oplus T_4 \cong \mathbb{C}^{10}$, hence $\dim \text{coker } \text{ob}_Y = 10$ and we are done. \square

Corollary 3.5. *We have*

$$\dim \text{Ext}^1(\Omega_Y^1, \mathcal{O}_Y) = 18.$$

Proof. Immediate from Corollary 3.2, Proposition 3.4 and exact sequence (11). \square

Proposition 3.6. *The following holds:*

- (i) $\text{Def}(Y)$ is smooth at $[Y]$, of dimension 18;
- (ii) $\text{ESDef}(Y)$ is smooth at $[Y]$, of dimension 8.

Proof. By Remark 3.3 and Corollary 3.5 we have

$$18 = \dim \text{Ext}^1(\Omega_Y^1, \mathcal{O}_Y) = \dim T_{[Y]} \text{Def}(Y) \geq \dim_{[Y]} \text{Def}(Y) \geq 18,$$

which proves (i).

On the other hand, if we move the branch loci $B_i \subset E_i$ the curve $\Delta \subset E_1 \times E_2$ remains of product type, so in this way we obtain a 8-dimensional family of *equisingular* deformations of Y ; therefore the equisingular deformation space $\text{ESDef}(Y)$ has dimension at least 8, and by Corollary 3.2 we have

$$8 = \dim H^1(\Theta_Y) = \dim T_{[Y]} \text{ESDef}(Y) \geq \dim_{[Y]} \text{ESDef}(Y) \geq 8.$$

This proves (ii). \square

Summing up, Proposition 3.6 shows that the deformations of Y are unobstructed and that they are all obtained by deforming the pair (A, Δ) , where A is an abelian surface and Δ a polarization of type $(4, 4)$. In particular, all the deformations preserve the action of H . Moreover, the equisingular deformations of Y are also unobstructed and are obtained by taking as A the product of two elliptic curves and by choosing the polarization Δ of product type.

Remark 3.7. *Since Y has only rational double points, by [BW74] the dimension of $\text{Def}(Y)$ equals the dimension of $\text{Def}(V)$. Then*

$$24 = h^1(\Theta_V) = \dim T_{[V]} \text{Def}(V) > \dim_{[V]} \text{Def}(V) = 18,$$

that is $\text{Def}(V)$ is singular at $[V]$. By [BW74, Theorem 3.7], this means that the sixteen (-2) -curves of V do not have independent behavior in deformations.

4. DEFORMATIONS OF THE SINGULAR PRODUCT-QUOTIENT SURFACE $X = Z/G$

Let us consider now the surface $X = Z/G$ defined in Section 2 and its minimal resolution of singularities $\lambda: S \rightarrow X$. We must analyze several cases, according to the type of quotient singularities that X contains.

Throughout this section we set $Q := \mathbb{P}^1 \times \mathbb{P}^1$ and we denote by $\mathcal{O}_Q(a, b)$ the line bundle of bidegree (a, b) on Q .

The following exact sequence is the analogue of (11):

$$(14) \quad 0 \rightarrow H^1(\Theta_X) \rightarrow \text{Ext}^1(\Omega_X^1, \mathcal{O}_X) \rightarrow \mathcal{T}_X^1 \xrightarrow{\text{ob}_X} H^2(\Theta_X).$$

4.1. Example where $\text{Sing}(X) = 16 \times \frac{1}{4}(1, 3)$. Assume that, locally around each of the fixed points, the action of $G = \langle \zeta \mid \zeta^4 = 1 \rangle$ is given by $\zeta \cdot (x, y) = (\zeta x, \zeta^{-1}y)$. Therefore,

$$\text{Sing}(X) = 16 \times \frac{1}{4}(1, 3).$$

In this case X contains only rational double points and we obtain

$$p_g(S) = 5, \quad q(S) = 0, \quad K_S^2 = 8.$$

Proposition 4.1. *S is a minimal surface of general type.*

Proof. S is of general type because $p_g(S) > 0$ and $K_S^2 > 0$. Since the action of G is twisted on the second factor and X has only rational double points, the Künneth formula and the third equality in (6) give

$$\begin{aligned} H^0(\omega_S^2) &= H^0(\omega_X^2) = H^0(\omega_Z^2)^G = H^0(\omega_{C_1}^2 \boxtimes \omega_{C_2}^2)^G \\ &= \bigoplus_{\chi \in \widehat{G}} (H^0(g_{1*}\omega_{C_1}^2)^\chi \otimes H^0(g_{2*}\omega_{C_2}^2)^\chi) = \mathbb{C}^{14}. \end{aligned}$$

This shows that $h^0(\omega_S^2) = K_S^2 + \chi(\mathcal{O}_S)$, hence S is a minimal surface. \square

Proposition 4.2. *The following holds:*

- (i) ob_X is surjective;
- (ii) $h^1(\Theta_X) = 2$, $h^2(\Theta_X) = 6$, $h^1(\Theta_S) = 50$, $h^2(\Theta_S) = 6$.
- (iii) $\text{ESDef}(X)$ is smooth at $[X]$, of dimension 2.

Proof. (i) Let us consider the dual map $\text{ob}_X^* : H^2(\Theta_X)^* \rightarrow (\mathcal{T}_X^1)^*$. By Grothendieck duality (see [AK70, Chapter I]) and Künneth formula we obtain

$$\begin{aligned}
 H^2(\Theta_X)^* &= H^0(\Omega_Z^1 \otimes \Omega_Z^2)^G \\
 &= \bigoplus_{\chi \in \widehat{G}} [(H^0(g_{1*}\omega_{C_1})^\chi \otimes H^0(g_{2*}\omega_{C_2}^2)^\chi) \\
 &\quad \oplus (H^0(g_{1*}\omega_{C_1}^2)^\chi \otimes H^0(g_{2*}\omega_{C_2})^\chi)] \\
 &= U_1 \oplus U_2, \text{ where} \\
 U_1 &= H^0(\omega_{\mathbb{P}^1} \otimes \mathcal{M}_1^2) \otimes H^0(\omega_{\mathbb{P}^1}^2(B_2) \otimes \mathcal{M}_2^2), \\
 U_2 &= H^0(\omega_{\mathbb{P}^1}^2(B_1) \otimes \mathcal{M}_1^2) \otimes H^0(\omega_{\mathbb{P}^1} \otimes \mathcal{M}_2^2).
 \end{aligned}
 \tag{15}$$

This yields $h^2(\Theta_X) = 6$ and so $h^2(\Theta_S) = 6$. Now we set

$$\begin{aligned}
 B_1 &= b'_1 + b'_2 + b'_3 + b'_4 \\
 B_2 &= b''_1 + b''_2 + b''_3 + b''_4
 \end{aligned}$$

and we choose local coordinates (x, y) in Z vanishing at (b'_i, b''_j) . As in Section 3, we can interpret ob_X^* as a map

$$\text{ob}_X^* : H^0(\Omega_Z^1 \otimes \Omega_Z^2)^G \rightarrow (p_*\Omega_Z^1)^G / \Omega_X^1,$$

where $(p_*\Omega_Z^1)^G / \Omega_X^1$ is a skyscraper sheaf supported on the singular points of X and locally generated by $x^i y^{i+1} dx - y^i x^{i+1} dy$, for $i = 0, 1, 2$, see [Cat89].

A straightforward local computation shows that the summand U_1 in (15) contributes expressions of the form $\alpha_1 \beta_1 x dy \otimes (dx \wedge dy)$ whereas the summand U_2 contributes expressions of the form $\alpha_2 \beta_2 y dx \otimes (dx \wedge dy)$, where $\alpha_i = \alpha_i(x^2)$ and $\beta_i = \beta_i(y^2)$ are pullbacks of local functions on \mathbb{P}^1 . Therefore the map ob_X^* can be identified, up to a multiplicative constant, with

$$\begin{aligned}
 \phi : H^0(\omega_{\mathbb{P}^1}^2(B_1) \otimes \mathcal{M}_1^2) \oplus H^0(\omega_{\mathbb{P}^1}^2(B_2) \otimes \mathcal{M}_2^2) \\
 \rightarrow \bigoplus_{i,j=1}^4 \mathbb{C}_{ij} \subset \bigoplus_{i,j=1}^4 \mathbb{C}_{ij}^{\oplus 3} \cong (\mathcal{T}_X^1)^* \\
 \phi(\sigma \oplus \tau) = \bigoplus_{i,j=1}^4 (\text{val}_{b'_i}(\sigma) - \text{val}_{b''_j}(\tau)),
 \end{aligned}$$

where the valuation maps are defined as in Section 3. Hence we obtain

$$\begin{aligned}
 \ker \phi &= \{\sigma \oplus \tau \mid \text{val}_{b'_1}(\sigma) = \text{val}_{b'_2}(\sigma) = \text{val}_{b'_3}(\sigma) = \text{val}_{b'_4}(\sigma) \\
 &= \text{val}_{b''_1}(\tau) = \text{val}_{b''_2}(\tau) = \text{val}_{b''_3}(\tau) = \text{val}_{b''_4}(\tau)\}.
 \end{aligned}
 \tag{16}$$

On the other hand, the valuation map $H^0(\omega_{\mathbb{P}^1}^2(B_i) \otimes \mathcal{M}_i^2) \rightarrow H^0(N_{B_i})$ can be identified with the residue map $H^0(\omega_{\mathbb{P}^1}(B_i)) \rightarrow H^0(N_{B_i})$ via the isomorphism $H^0(\omega_{\mathbb{P}^1}^2(B_i) \otimes \mathcal{M}_i^2) \cong H^0(\omega_{\mathbb{P}^1}(B_i))$. By the Residue Theorem we have

$$\sum_{i=1}^4 \text{val}_{b'_i}(\sigma) = \sum_{j=1}^4 \text{val}_{b''_j}(\tau) = 0,$$

so (16) implies that $\sigma \oplus \tau \in \ker \phi$ if and only if $\text{val}_{b'_i}(\sigma) = \text{val}_{b''_j}(\tau) = 0$ for all pairs (i, j) . But there are no non-zero holomorphic 1-forms on \mathbb{P}^1 ,

so $\ker \phi = 0$ and ob_X^* is injective. Therefore the obstruction map ob_X is surjective.

(ii) Let us denote by $F \subset S$ the exceptional divisor of $\lambda: S \rightarrow X$. Since S has only rational double points, we have

$$H^1(\Theta_S) \cong H^1(\Theta_X) \oplus H_F^1(\Theta_S), \quad H^2(\Theta_S) \cong H^2(\Theta_X).$$

By Riemann-Roch theorem we obtain

$$h^1(\Theta_S) - h^2(\Theta_S) = 10\chi(\mathcal{O}_S) - 2K_S^2 = 44,$$

then $h^1(\Theta_S) = 50$ since we have shown that $h^2(\Theta_S) = 6$, see part (i). Being F the union of sixteen disjoint A_3 -cycles, we have $H_F^1(\Theta_S) \cong \mathbb{C}^{16 \cdot 3} = \mathbb{C}^{48}$. Therefore $h^1(\Theta_X) = 2$.

(iii) The cover $u: X \rightarrow Q$ is a simple G -cover branched on the divisor $B = B_1 \times B_2$, which has bidegree $(4, 4)$. By varying the branch loci $B_i \subset \mathbb{P}^1$ we obtain a 2-dimensional family of equisingular deformations of X . Then

$$2 = \dim H^1(\Theta_X) = \dim T_{[X]} \text{ESDef}(X) \geq \dim_{[X]} \text{ESDef}(X) \geq 2,$$

which implies the claim. \square

Proposition 4.3. *The general deformation of the surface X is a canonically embedded, smooth complete intersection $S_{2,4}$ of type $(2, 4)$ in \mathbb{P}^4 .*

Proof. By [Cat97, Proposition 6.2] it is sufficient to check that the canonical map $\phi_K: X \rightarrow \mathbb{P}^4$ is a birational morphism onto its image. Since X has only Rational Double Points and $u: X \rightarrow Q$ is a *simple* G -cover, Hurwitz formula yields $K_X = u^*\mathcal{O}_Q(1, 1)$; but $|\mathcal{O}_Q(1, 1)|$ is base-point free, so $|K_X|$ is also base-point free and ϕ_K is a morphism.

It remains to show that ϕ_K separates two general points x, y on X . The decomposition of $u_*\omega_X$ with respect to the G -action is

$$u_*\omega_X = \omega_Q \oplus (\omega_Q \otimes L) \oplus (\omega_Q \otimes L^2) \oplus (\omega_Q \otimes L^3),$$

where $L = \mathcal{O}_Q(1, 1)$ and $\omega_Q \otimes L^i$ is the eigensheaf corresponding to the character χ_i . Therefore we obtain

$$H^0(u_*\omega_X) = H^0(\omega_Q \otimes L^2) \oplus H^0(\omega_Q \otimes L^3).$$

Now let $\{\tau\}$ be a basis of $H^0(\omega_Q \otimes L^2) = H^0(\mathcal{O}_Q)$ and let $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ be a basis of $H^0(\omega_Q \otimes L^3) = H^0(\mathcal{O}_Q(1, 1))$. The four sections $\{\sigma_i\}$ provide an embedding $Q \hookrightarrow \mathbb{P}^3$, hence ϕ_K separates pairs of points which belong to the same fibre of $u: X \rightarrow Q$. Now let x, y be two points in the same (general) fibre of u . Then there exists $1 \leq a \leq 3$ such that $y = \zeta^a \cdot x$. Then

$$\sigma_i(y) = \zeta^a \sigma_i(x), \quad \tau(y) = \zeta^{2a} \tau(x),$$

that is

$$\begin{aligned} \phi_K(y) &= [\sigma_1(y): \sigma_2(y): \sigma_3(y): \sigma_4(y): \tau(y)] \\ &= [\sigma_1(x): \sigma_2(x): \sigma_3(x): \sigma_4(x): \zeta^a \tau(x)] \\ &\neq [\sigma_1(x): \sigma_2(x): \sigma_3(x): \sigma_4(x): \tau(x)] = \phi_K(x). \end{aligned}$$

Therefore ϕ_K also separates general pairs of points lying in the same fibre of $u: X \rightarrow Q$ and we are done. \square

Now we can prove the following

Proposition 4.4. *Def(X) is smooth at $[X]$, of dimension 44.*

Proof. By using Proposition 4.2 and exact sequence (14) we obtain

$$(17) \quad \dim T_{[X]} \text{Def}(X) = \dim \text{Ext}^1(\Omega_X^1, \mathcal{O}_X) = 44.$$

On the other hand, by [Se06, Chapter 3] one knows that $\text{Def}(S_{2,4})$ is smooth, of dimension

$$h^0(N_{S_{2,4}/\mathbb{P}^4}) - \dim \text{Aut}(\mathbb{P}^4) = h^0(\mathcal{O}_{S_{2,4}}(2)) + h^0(\mathcal{O}_{S_{2,4}}(4)) - 24 = 44.$$

Equality (17) and Proposition 4.3 yield

$$(18) \quad 44 = \dim T_{[X]} \text{Def}(X) \geq \dim_{[X]} \text{Def}(X) = \dim_{[S_{2,4}]} \text{Def}(S_{2,4}) = 44,$$

so we are done. \square

Remark 4.5. *Since X has only rational double points, by [BW74] the dimension of $\text{Def}(X)$ equals the dimension of $\text{Def}(S)$. So we infer*

$$50 = h^1(\Theta_S) = \dim T_{[S]} \text{Def}(S) > \dim_{[S]} \text{Def}(S) = 44,$$

that is $\text{Def}(S)$ is singular at $[S]$. By [BW74, Theorem 3.7], this means that the sixteen A_3 -cycles of S do not have independent behavior in deformations.

Proposition 4.3 in particular shows that the general deformation of X does not preserve the G -action. Now we want to consider some particular deformations that preserve the quadruple cover $u: X \rightarrow Q$. According to [Pa91] we call them *natural deformations*, and we freely follow the notation of that paper everywhere. The building data of any totally ramified G -cover $u: X \rightarrow Q$ are

$$(19) \quad \begin{aligned} 4L_{\chi_1} &= 3D_{G,\chi_3} + D_{G,\chi_1} \\ 2L_{\chi_2} &= D_{G,\chi_1} + D_{G,\chi_3} \\ 4L_{\chi_3} &= D_{G,\chi_3} + 3D_{G,\chi_1}, \end{aligned}$$

see [Pa91, Proposition 2.1]. The G -cover $u: X \rightarrow Q$ defines a natural embedding i of X into the total space of the vector bundle $W = \bigoplus_{\chi \in \widehat{G} \setminus \{\chi_0\}} V(L_\chi^{-1})$.

If w_χ is a local coordinate on $V(L_\chi^{-1})$ on an open set U and $\sigma_{G,\psi}$ is a local equation for $D_{G,\psi}$ on U , then $i(X)$ is defined by the equations

$$(20) \quad w_\chi w_{\chi'} = \left(\prod_{\psi \in \{\chi_1, \chi_3\}} (\sigma_{G,\psi})^{\epsilon_{\chi,\chi'}^{G,\psi}} \right) w_{\chi\chi'}$$

and the covering map is given by the composition $\pi \circ i$, where $\pi: W \rightarrow Q$ is the projection. Moreover, the integers $\epsilon_{\chi,\chi'}^{G,\psi}$ can be easily computed by using [Pa91, p. 196]:

$$(21) \quad \begin{aligned} \epsilon_{\chi_0,\chi_0}^{G,\chi_1} &= 0, & \epsilon_{\chi_0,\chi_1}^{G,\chi_1} &= 0, & \epsilon_{\chi_0,\chi_2}^{G,\chi_1} &= 0, & \epsilon_{\chi_0,\chi_3}^{G,\chi_1} &= 0, & \epsilon_{\chi_1,\chi_1}^{G,\chi_1} &= 0, \\ \epsilon_{\chi_1,\chi_2}^{G,\chi_1} &= 0, & \epsilon_{\chi_1,\chi_3}^{G,\chi_1} &= 1, & \epsilon_{\chi_2,\chi_2}^{G,\chi_1} &= 1, & \epsilon_{\chi_2,\chi_3}^{G,\chi_1} &= 1, & \epsilon_{\chi_3,\chi_3}^{G,\chi_1} &= 1, \\ \epsilon_{\chi_0,\chi_0}^{G,\chi_3} &= 0, & \epsilon_{\chi_0,\chi_1}^{G,\chi_3} &= 0, & \epsilon_{\chi_0,\chi_2}^{G,\chi_3} &= 0, & \epsilon_{\chi_0,\chi_3}^{G,\chi_3} &= 0, & \epsilon_{\chi_1,\chi_1}^{G,\chi_3} &= 1, \\ \epsilon_{\chi_1,\chi_2}^{G,\chi_3} &= 1, & \epsilon_{\chi_1,\chi_3}^{G,\chi_3} &= 1, & \epsilon_{\chi_2,\chi_2}^{G,\chi_3} &= 1, & \epsilon_{\chi_2,\chi_3}^{G,\chi_3} &= 0, & \epsilon_{\chi_3,\chi_3}^{G,\chi_3} &= 0. \end{aligned}$$

Let us consider now a collection of sections

$$\{r_{G,\psi,\chi} \in H^0(\mathcal{O}_Q(D_{G,\psi}) \otimes L_\chi^{-1})\}_{\psi \in \{\chi_1, \chi_3\}, \chi \in S_{G,\psi}},$$

where

$$S_{G,\chi_1} := \{\chi_0, \chi_1, \chi_2\}, \quad S_{G,\chi_3} := \{\chi_0, \chi_2, \chi_3\}.$$

Let $h_{G,\psi,\chi}$ be a local representative of $r_{G,\psi,\chi}$ on the open set U and define

$$\tau_{G,\psi} := \sum_{\substack{\psi \in \{\chi_1, \chi_3\} \\ \chi \in S_{G,\psi}}} h_{G,\psi,\chi} w_\chi.$$

Then the natural deformation of the G -cover $u: X \rightarrow Q$, associated to the collection of sections $\{r_{G,\psi,\chi}\}$, is the subvariety X' of W locally defined by

$$w_\chi w_{\chi'} = \left(\prod_{\psi \in \{\chi_1, \chi_3\}} (\tau_{G,\psi})^{\epsilon_{\chi,\chi'}^{G,\psi}} \right) w_{\chi\chi'},$$

together with the map $u': X' \rightarrow Q$ obtained by restricting the projection $\pi: W \rightarrow Q$ to X' .

Coming back to our particular case, we have

$$D_{G,\chi_1} \in |\mathcal{O}_Q(4, 4)|, \quad D_{G,\chi_3} = 0,$$

$$L_{\chi_1} \cong \mathcal{O}_Q(1, 1), \quad L_{\chi_2} \cong \mathcal{O}_Q(2, 2), \quad L_{\chi_3} \cong \mathcal{O}_Q(3, 3),$$

and $B = D_{G,\chi_1}$. Since $D_{G,\chi_3} = 0$, the natural deformations of X are parameterized by the vector space

$$\begin{aligned} (22) \quad & \bigoplus_{\chi \in S_{G,\chi_1}} H^0(\mathcal{O}_Q(D_{G,\chi_1}) \otimes L_\chi^{-1}) \\ &= H^0(\mathcal{O}_Q(4, 4)) \oplus H^0(\mathcal{O}_Q(3, 3)) \oplus H^0(\mathcal{O}_Q(2, 2)) \cong \mathbb{C}^{50}. \end{aligned}$$

4.2. Example where $\text{Sing}(X) = 16 \times \frac{1}{4}(1, 1)$. Assume that, locally around each of the fixed points, the action of $G = \langle \zeta \mid \zeta^4 = 1 \rangle$ is given by $\zeta \cdot (x, y) = (\zeta x, \zeta y)$. In this case,

$$\text{Sing}(X) = 16 \times \frac{1}{4}(1, 1).$$

By using Proposition 1.3, we obtain

$$p_g(S) = 1, \quad q(S) = 0, \quad K_S^2 = -8,$$

hence S is not a minimal model.

Theorem 4.6. *The following holds:*

- (i) $h^2(\Theta_X) = 14$;
- (ii) all natural deformations of $u: X \rightarrow Q$ preserve the 16 points of type $\frac{1}{4}(1, 1)$;
- (iii) there exists a 12-dimensional family of \mathbb{Q} -Gorenstein deformations of X , smoothing all the singularities. The general element X_t of this deformation is a smooth, minimal surface of general type with $p_g(X_t) = 1$, $q(X_t) = 0$ and $K_{X_t}^2 = 8$;
- (iv) X_t is isomorphic to a Todorov surface with $K^2 = 8$.

Proof. (i) By using Grothendieck duality and Künneth formula as in Proposition 4.2 we obtain

$$\begin{aligned}
H^2(\Theta_X)^* &= H^0(\Omega_Z^1 \otimes \Omega_Z^2)^G \\
&= \bigoplus_{\chi \in \widehat{G}} [(H^0(g_{1*}\omega_{C_1})^\chi \otimes H^0(g_{2*}\omega_{C_2}^2)^{\chi^{-1}}) \\
&\quad \oplus (H^0(g_{1*}\omega_{C_1}^2)^\chi \otimes H^0(g_{2*}\omega_{C_2})^{\chi^{-1}})] \\
&= (H^0(\mathcal{O}_{\mathbb{P}^1}) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(2))) \oplus (H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1))) \\
&\quad \oplus (H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1))) \oplus (H^0(\mathcal{O}_{\mathbb{P}^1}(2)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1})),
\end{aligned}$$

which yields $h^2(\Theta_X) = 14$.

(ii) The G -cover $u: X \rightarrow Q$ is determined by the building data (19), with

$$D_{G,\chi_1} \in |\mathcal{O}_Q(4, 0)|, \quad D_{G,\chi_3} \in |\mathcal{O}_Q(0, 4)|,$$

$$L_{\chi_1} \cong \mathcal{O}_Q(1, 3), \quad L_{\chi_2} \cong \mathcal{O}_Q(2, 2), \quad L_{\chi_3} \cong \mathcal{O}_Q(3, 1).$$

The natural deformations of u are parameterized by the vector space

$$\begin{aligned}
(23) \quad & \bigoplus_{\psi \in \{\chi_1, \chi_3\}} \left(\bigoplus_{\chi \in S_{G,\psi}} H^0(\mathcal{O}_Q(D_{G,\psi}) \otimes L_\chi^{-1}) \right) \\
&= H^0(\mathcal{O}_Q(4, 0)) \oplus H^0(\mathcal{O}_Q(0, 4)).
\end{aligned}$$

Therefore they form a family of dimension 10, which is exactly the one obtained by keeping the branch divisor $B \subset Q$ of product type. In particular, all the natural deformations preserve the sixteen singular points of X .

(iii) For simplicity, set $w_i = w_{\chi_i}$ and $\tau_{G,\chi_i} = h_i w_0$. Writing $w_0 = 1$, the local equations defining the family of natural deformations of $u: X \rightarrow Q$ are the following:

$$\begin{aligned}
(24) \quad & w_1^2 = h_3 w_2, \quad w_1 w_2 = h_3 w_3, \quad w_1 w_3 = h_1 h_3, \\
& w_2^2 = h_1 h_3, \quad w_2 w_3 = h_1 w_1, \quad w_3^2 = h_1 w_2.
\end{aligned}$$

Relations (24) can be written in determinantal form in two different ways, namely

$$\begin{aligned}
(\mathbf{a}) \quad & \text{rank} \begin{pmatrix} w_2 & w_3 & w_1 & h_1 \\ w_1 & w_2 & h_3 & w_3 \end{pmatrix} \leq 1, \\
(\mathbf{b}) \quad & \text{rank} \begin{pmatrix} h_3 & w_1 & w_2 \\ w_1 & w_2 & w_3 \\ w_2 & w_3 & h_1 \end{pmatrix} \leq 1.
\end{aligned}$$

In the sequel we will only consider the determinantal representation (b). We can deform it by using the parameter $s \in H^0(L_{\chi_2}) = \mathbb{C}^9$, i.e.

$$(25) \quad \text{rank} \begin{pmatrix} h_3 & w_1 & w_2 \\ w_1 & w_2 + s & w_3 \\ w_2 & w_3 & h_1 \end{pmatrix} \leq 1.$$

It is no difficult to check that for general $s \neq 0$ one obtains a smooth surface, hence (25) provides a smoothing $\pi: \mathcal{X} \rightarrow T$ of X . This is actually a \mathbb{Q} -Gorenstein smoothing of X , since it is the globalization of the local \mathbb{Q} -Gorenstein smoothing of the quotient singularity $\frac{1}{4}(1, 1)$, see [Man08,

Chapter 4]. Therefore the general fibre X_t of π is a surface of general type whose invariants are

$$p_g(X_t) = 1, \quad q(X_t) = 0, \quad K_{X_t}^2 = 8.$$

The canonical divisor K_X is big and nef (since $4K_X = u^*\mathcal{O}_Q(4, 4)$), so K_{X_t} is big and nef too, as X_t is obtained by a \mathbb{Q} -Gorenstein smoothing of X . This shows that X_t is a minimal model.

In order to give a more concrete description of X_t , let us look again at the double cover $v: Y \rightarrow E_1 \times E_2$ constructed in Section 3. By Proposition 3.6 we know that $\text{Def}(Y)$ is smooth at $[Y]$ of dimension 18; moreover the general deformation Y_t of Y is a double cover $v_t: Y_t \rightarrow A_t$ of an abelian variety A_t , branched on a smooth divisor Ξ which is a polarization of type $(4, 4)$. Let us compute the dimension of the subspace of $\text{Def}(Y)$ consisting of surfaces for which it is possible to lift the natural involution $\iota_t: A_t \rightarrow A_t$ to an involution $\tilde{\iota}_t: Y_t \rightarrow Y_t$ such that $Y_t/\tilde{\iota}_t$ is smooth. By [BL04, Corollary 4.7.6], the divisor Ξ does not contain any of the 16 fixed points of ι_t . If we write locally the equation of the double cover $v_t: Y_t \rightarrow A_t$ as $z^2 = f(x, y)$ so that ι_t is given by $(x, y) \rightarrow (-x, -y)$, we see that ι_t lifts to Y_t if and only if the branch locus $f(x, y) = 0$ is ι_t -invariant; moreover in this case there is a unique lifting such that the quotient is smooth; it is locally given by $(x, y, z) \rightarrow (-x, -y, -z)$. By [BL04, Corollary 4.6.6], the divisors in $|\Xi|$ which are invariant under ι_t form a family of dimension $\frac{1}{2}h^0(\mathcal{O}_A(\Xi)) + 2 - 1 = 9$ and so, taking into account the three moduli of abelian surfaces, we obtain a 12-dimensional family $\{Y_t\}$ of deformations of Y which admit a lifting of ι_t .

One can further check that the lifted involution $\tilde{\iota}$ is fixed-point free and that the family $\{X_t\}$ constructed before can be obtained as $X_t = Y_t/\tilde{\iota}_t$.

(*iv*) Let us consider the Kummer surface $\text{Kum}(A_t) := A_t/\iota_t$. By (*iii*) a general fibre X_t of the \mathbb{Q} -Gorenstein smoothing of X is a double cover of $\text{Kum}(A_t)$ branched over the 16 nodes of $\text{Kum}(A_t)$ and the image D of the curve Ξ .

On the other hand, $\text{Kum}(A_t)$ can be embedded in \mathbb{P}^3 as a quartic surface with 16 nodes and via this embedding the curve D is obtained by intersecting $\text{Kum}(A_t)$ with a smooth quadric surface Φ which does not contain any of the nodes.

This shows that X_t belongs precisely to the family of surfaces with $p_g = 1$, $q = 0$ and $K^2 = 8$ constructed by Todorov in [To81]. \square

Remark 4.7. *Let us fix the abelian surface A and the embedding $\text{Kum}(A) \hookrightarrow \mathbb{P}^3$. Then the choice of the deformation parameter $s \in H^0(L_{X_2})$ corresponds to the choice of the quadric surface $\Phi \in |\mathcal{O}_{\mathbb{P}^3}(2)|$. By [To81, Lemma 2.1] there is a quadric surface Φ_k in \mathbb{P}^3 which contains exactly k ($1 \leq k \leq 6$) of the nodes of $\text{Kum}(A)$ that are general position. This means that the pullback in A of the curve $D_k := \text{Kum}(A) \cap \Phi_k$ is a polarization of type $(4, 4)$ which contains exactly k of the fixed points of $\iota: A \rightarrow A$.*

Therefore arguments similar to those used in the proof of Theorem 4.6, part (*ii*) show that there exists a partial \mathbb{Q} -Gorenstein smoothing of X , whose general fibre X_t is isomorphic to the double cover of $\text{Kum}(A)$ branched over the curve D_k and the remaining $16 - k$ nodes of $\text{Kum}(A)$. The surface X_t is not smooth, since it contains exactly k singular points of type $\frac{1}{4}(1, 1)$.

Its minimal resolution of singularities is a Todorov surface with $K^2 = 8 - k$ ($1 \leq k \leq 6$).

4.3. Example where $\text{Sing}(X) = 8 \times \frac{1}{4}(1, 3) + 8 \times \frac{1}{4}(1, 1)$. We can also twist the action of G on Z in such a way that

$$\text{Sing}(X) = 8 \times \frac{1}{4}(1, 1) + 8 \times \frac{1}{4}(1, 3).$$

By using Proposition 1.3, we obtain

$$p_g(S) = 3, \quad q(S) = 0, \quad K_S^2 = 0,$$

hence S is not a minimal model.

Rasdeaconu and Suvaina give an explicit construction of S in [RS06, Section 3], showing that it is a simply connected, minimal, elliptic surface with no multiple fibers. One can also prove that $H^2(\Theta_X) \neq 0$, see [LP11, Section 3].

Proposition 4.8. *The following holds:*

- (i) *all natural deformations of X preserve the 8 points of type $\frac{1}{4}(1, 1)$;*
- (ii) *there exists a family of \mathbb{Q} -Gorenstein deformations of X , smoothing all the singularities. The general element of this family is a smooth, minimal surface of general type with $p_g = 3$, $q = 0$ and $K^2 = 8$.*

Proof. (i) The abelian G -cover $u: X \rightarrow Q$ is determined by the building data (19), with

$$\begin{aligned} D_{G, \chi_1}, D_{G, \chi_3} &\in |\mathcal{O}_Q(2, 2)|. \\ L_{\chi_1}, L_{\chi_2}, L_{\chi_3} &\cong \mathcal{O}_Q(2, 2). \end{aligned}$$

The same argument of Theorem 4.6, part (ii) shows that the natural deformations of X are parameterized by the vector space

$$\begin{aligned} &H^0(\mathcal{O}_Q(2, 2)) \oplus H^0(\mathcal{O}_Q(2, 2)) \\ &\oplus H^0(\mathcal{O}_Q) \oplus H^0(\mathcal{O}_Q) \oplus H^0(\mathcal{O}_Q) \oplus H^0(\mathcal{O}_Q). \end{aligned}$$

Writing $w_i := w_{\chi_i}$ we have

$$h_1 = g_1 + c_1 w_1 + c_2 w_2, \quad h_3 = g_3 + d_2 w_2 + d_3 w_3,$$

where g_i a local equations of D_{G, χ_i} and $c_i, d_i \in \mathbb{C}$. Therefore the equations of the natural deformations of X are

$$\begin{aligned} (26) \quad &w_1^2 = (g_3 + d_2 w_2 + d_3 w_3)w_2, \\ &w_1 w_2 = (g_3 + d_2 w_2 + d_3 w_3)w_3, \\ &w_1 w_3 = (g_1 + c_1 w_1 + c_2 w_2)(g_3 + d_2 w_2 + d_3 w_3), \\ &w_2^2 = (g_1 + c_1 w_1 + c_2 w_2)(g_3 + d_2 w_2 + d_3 w_3), \\ &w_2 w_3 = (g_1 + c_1 w_1 + c_2 w_2)w_1, \\ &w_3^2 = (g_1 + c_1 w_1 + c_2 w_2)w_2. \end{aligned}$$

For a general choice of the parameters the morphism $\bar{u}: \bar{X} \rightarrow Q$ is *not* a Galois cover and an easy computation shows that its branch locus is of the form

$$D_{\bar{X}} = D_1 + \dots + D_6$$

where the D_i belong to the pencil generated by D_{G,χ_1} and D_{G,χ_3} . Then the singular locus of $D_{\bar{X}}$ is given by the 8 points $D_{G,\chi_1} \cap D_{G,\chi_3}$ and $\text{Sing}(\bar{X})$ consists of the 8 points of type $\frac{1}{4}(1, 1)$ locally defined by setting

$$g_1 = g_3 = w_1 = w_2 = w_3 = 0$$

in (26).

(ii) We note that the set of natural deformations \bar{X} of X which keep the G -action is parameterized by the vector space $H^0(\mathcal{O}_Q(2, 2)) \oplus H^0(\mathcal{O}_Q(2, 2))$. In fact, the action of the generator $i = \sqrt{-1}$ of G must be given by

$$w_1 \mapsto -iw_1, \quad w_2 \mapsto -w_2, \quad w_3 \mapsto iw_3$$

and substituting in (26) we obtain $c_1 = c_2 = d_1 = d_3 = 0$.

The G -cover $\bar{X} \rightarrow Q$ factors into two double covers

$$\bar{X} \rightarrow K \xrightarrow{p} Q$$

where K is a $K3$ surface with 8 ordinary double points and $p: K \rightarrow Q$ is a double cover branched over $D_{G,\chi_1} + D_{G,\chi_3}$. Let D_{G,χ_2} be a general member in the pencil induced by D_{G,χ_1} and D_{G,χ_3} . Let $\bar{D}_{G,\chi_2} = p^*D_{G,\chi_2}$ and $2\bar{D}_{G,\chi_i} = p^*D_{G,\chi_i}$ for $i = 1, 3$. Since D_{G,χ_2} is linearly equivalent to D_{G,χ_i} for $i = 1, 3$ and a $K3$ surface is simply connected, \bar{D}_{G,χ_2} is linearly equivalent to $\bar{D}_{G,\chi_1} + \bar{D}_{G,\chi_3}$. Note that both these curves have exactly 8 nodes. The double cover \tilde{X} of K branched over \bar{D}_{G,χ_2} is deformation equivalent to \bar{X} , and \tilde{X} can be realized as the bidouble cover of Q branched over D_{G,χ_1} , D_{G,χ_3} and D_{G,χ_2} . Therefore if one deforms D_{G,χ_2} to a general divisor of bidegree $(2, 2)$ we have a \mathbb{Q} -Gorenstein smoothing of \tilde{X} which smoothes all the singularities. Since \bar{X} is a deformation of X and \tilde{X} is deformation equivalent to \bar{X} , we have a smooth projective surface in the deformation space of X which is a \mathbb{Q} -Gorenstein smoothing of \bar{X} . Finally, we note that each deformation is a \mathbb{Q} -Gorenstein one. In fact, \tilde{X} and \bar{X} are double covers of the $K3$ surface K branched over \bar{D}_{G,χ_2} and $\bar{D}_{G,\chi_1} + \bar{D}_{G,\chi_3}$, respectively. Let $\mathcal{X} \rightarrow \Delta$ be a family of double covers of K obtained deforming the branch locus from $\bar{D}_{G,\chi_1} + \bar{D}_{G,\chi_3}$ to \bar{D}_{G,χ_2} . By using the canonical divisor formula for a double cover, it is not hard to see that $K_{\mathcal{X}}$ is a \mathbb{Q} -Cartier divisor. Therefore the transitive property of \mathbb{Q} -Gorenstein deformations implies that X has a \mathbb{Q} -Gorenstein smoothing. \square

Remark 4.9. *By applying arguments similar to those used in Remark 4.7 and in [Lee10, Section 2], one can construct surfaces of general type with $p_g = 3$, $q = 0$ and $K^2 = k$ ($2 \leq k \leq 8$) by first taking a \mathbb{Q} -Gorenstein smoothing of k singular points of type $\frac{1}{4}(1, 1)$ of \bar{X} and then the minimal resolution of the remaining $8 - k$ singular points of the same type.*

REFERENCES

- [AK70] A. Altman, S. Kleiman, Introduction to Grothendieck duality theory, Springer Lecture Notes in Mathematics **146** (1970).
- [BC04] I. Bauer and F. Catanese, Some new surfaces with $p_g = q = 0$, The Fano Conference, 123–142, Univ. Torino, Turin (2004).
- [BCG08] I. Bauer, F. Catanese, and F. Grunewald, The classification of surfaces with $p_g = q = 0$ isogenous to a product of curves. Pure Appl. Math. Q. **4** (2008), 547–586.

- [BCGP] I. Bauer, F. Catanese, F. Grunewald, and R. Pignatelli, Quotients of products of curves, new surfaces with $p_g = 0$ and their fundamental groups, math.AG (math.GR).arXiv:0809.3420, to appear on American Journal of Mathematics.
- [BP] I. Bauer and R. Pignatelli, The classification of minimal product-quotient surfaces with $p_g = 0$, math.AG (math.GR).arXiv:1006.3209, to appear on Mathematics of Computation.
- [BL04] C. Birkenhake and H. Lange, Complex abelian varieties. 2nd ed. Grundlehren der Mathematischen Wissenschaften, 302. Springer-Verlag, Berlin, 2004.
- [BW74] D. Burns and J. Wahl, Local contributions to global deformations of surfaces, Invent. Math. **26** (1974), 67–88.
- [CP09] G. Carnovale and F. Polizzi, The classification of surfaces with $p_g = q = 1$ isogenous to a product of curves, Adv. Geom. **9** (2009), 233–256.
- [Cat89] F. Catanese, Everywhere non reduced moduli spaces, Invent. Math. **98** (1989), 293–310.
- [Cat97] F. Catanese, Homological algebra and algebraic surfaces, Proceedings of Symposia in Pure Mathematics, Volume **62** (1997)
- [CD89] F. Catanese and O. Debarre, Surfaces with $K^2 = 2, p_g = 1, q = 0$, J. Reine Angew. Math. **395** (1989), 1–55.
- [KSB88] J. Kollár and Shepherd-Barron, Threefolds and deformations of surface singularities, Invent. Math. **91** (1988), 299–338.
- [Lau71] H. Laufer, Normal two-dimensional singularities, Annals of Mathematics Studies **71**, Princeton University Press 1971.
- [Lee10] Y. Lee, Complex structure on the rational blowdown of sections in $E(4)$, Algebraic Geometry in East Asia, Seoul 2008, 259–269, Adv. Stud. Pure Math., **60**, Math. Soc. Japan, Tokyo, 2010.
- [LP07] Y. Lee and J. Park, A simply connected surface of general type with $p_g = 0$ and $K^2 = 2$, Invent. Math. **170** (2007), 483–505.
- [LP11] Y. Lee and J. Park, A construction of Horikawa surface via \mathbb{Q} -Gorenstein smoothings, Math. Z. **267** (2011), 15–25.
- [LN11] Y. Lee and N. Nakayama, Simply connected surfaces of general type in positive characteristic via deformation theory, math.AG.arXiv:1103.5185.
- [Man08] M. Manetti, Smoothings of singularities and deformation types of surfaces, Symplectic 4-manifolds and algebraic surfaces, Lecture Notes in Mathematics **1938** (2008), 169–230.
- [MP10] E. Mistretta and F. Polizzi, Standard isotrivial fibrations with $p_g = q = 1$. II, J. Pure Appl. Algebra **214** (2010), 344–369.
- [Pa91] R. Pardini, Abelian covers of algebraic varieties, J. Reine angew. Math. **417** (1991), 191–213.
- [PPS1] H. Park, J. Park, and D. Shin, Surfaces of general type with $p_g = 1$ and $q = 0$, math.AG(math.GT).arXiv:0906.5195.
- [PPS2] H. Park, J. Park, and D. Shin, A simply connected surface of general type with $p_g = 1, q = 0$, and $K^2 = 8$, math.AG(math.GT).arXiv:0910.3506.
- [Pin81] H. Pinkham, Some local obstructions to deforming global surfaces, Nova Acta Leopold, New Folge **52** (1981), no. 240, 173–178.
- [Pol08] F. Polizzi, On surfaces of general type with $p_g = q = 1$ isogenous to a product of curves, Comm. Algebra **36** (2008), 2023–2053.
- [Pol09] F. Polizzi, Standard isotrivial fibrations with $p_g = q = 1$, J. Algebra **321** (2009), 1600–1631.
- [RS06] R. Rasdeaconu and I. Suvaina, The algebraic rational blow-down, math.SG (math.AG)/0601270.
- [Rito09] C. Rito, A note on Todorov surfaces, Osaka J. Math. **46** (2009), no. 3, 685–693.
- [Se06] E. Sernesi, Deformations of Algebraic Schemes, Grundlehren der Mathematischen Wissenschaften, **334**, Springer-Verlag, Berlin, 2006.
- [To81] A. Todorov, A construction of surfaces with $p_g = 1, q = 0$ and $2 \leq (K^2) \leq 8$. Counterexamples of the global Torelli theorem, Invent. Math. **63** (1981), 287–304.

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